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Alfred Kaufman

Center for Naval Analyses

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A modification of the Koopman framework for search and detection problems is proposed to account for nonuniform probabilities of target arrival. It essentially consists in determining various physical quantities relevant to the search and detection game for any of a family of simple target paths, each one of which is assigned a given probability of having been chosen from the family. The actual value of the physical quantity of interest is then obtained by averaging over the family parameters. (over)

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20 We develop this idea to provide probability densities of target arrival and detection probabilities in the search and detection environment. The mathematical structure of the problem is shown to reduce to a system of iterative equations that are easily amenable to numerical as well as analytical handling.

The most appealing quality of our formal structure is the capability it has to account for such realistic features of the search and detection game as partial knowledge of target track and correlations along the path of the searcher.

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INTRODUCTION

The years following the Second World War have seen a substantial development in understanding the search and detection game involving two military units searching for each other with some specified detection hardware. It has become apparent that Koopman's attempt - a good part of which stands canonized in his 1946 book - establishes a suitable framework for the description of these operations. Much of the early work, however, fails to adequately accommodate the modified rules of the game that technological development as well as increased analytical exigencies have since imposed upon us.

It is the contention of the author that notwithstanding the proliferation of papers that have been written on the subject attempting to account for the complexities of modern search and detection, a clear framework malleable enough to handle a large variety of outstanding real life problems has failed to emerge. Therefore, the purpose of this paper will be to revisit, criticize, slightly modify, and subsequently expand the original Koopman framework.

Perhaps it is worth noticing from the onset that we have chosen to address little, if anything, of this work to the question of comparison with the state of the art. We have done so because we are more interested in presenting an independent framework, whose right for survival we shall heretofore attempt to argue, rather than improve on limited treatments available in the literature.

1. THE KOOPMAN APPROACH REVISITED

In the original Koopman formulation, the game of search and detection includes two players, the searcher and a target. The two players engage in some sort of geometrical search pattern, whereby each one of them moves along deterministic trajectories. This kinematical part of the game is well described mathematically by the relative track of the target,

$$\xi = \xi(t) \quad ; \qquad \eta = \eta(t) \quad . \tag{I-1}$$

While moving on its trajectory the searcher is assumed to be looking, either through glimpses or continuously, for the target. The work "looking" is used here to mean attempt of detection via some hardware, be it visual, radar, sonar, or any other means whatsoever. This part of the game is purely probabilistic in nature and the central mathematical entity to be considered is the instantaneous probability density of detection $\gamma(t)$, where:

$$\gamma(t) = \gamma\left(\sqrt{\xi^2(t) + \eta^2(t)}\right) \quad . \tag{I-2}$$

Notice that in equation (I-2), $\gamma(t)$ is not taken to depend on the relative bearing of the target. While not a necessary assumption, it certainly is a very reasonable one for most detection hardware.

Once $\gamma(t)$ is given, the probability of detection for continuous looking through time t is given by:

$$p(t) = 1 - e^{-\int_0^t d\tau \gamma(\tau)} . \tag{I-3}$$

In principle, equation (I-3) provides the probability of detection for relative tracks of any given complexity. In practice, one is limited, of course, by one's capability of estimating the exponent in (I-3) along the track. In fact, the task of evaluating the integral in question is more often than not a quite formidable one, and correspondingly, solutions are difficult to get for all but a few instances which, in virtue of their simplicity, are hardly realistic. It is also evident, that a solution to equation (I-3) is less than realistic in that it fails to take into account the random nature of target arrival.

A somewhat more significant result obtains if one is willing to restrict the Koopman treatment by making the following strong assumptions:

- The target's position is uniformly distributed over some region of size A in the ocean.
- The searcher's path is random in A in the sense that it can be thought of as having its different (not too near) portions placed independently of one another in A.

Then, the probability of detection is given by the Koopman random search formula,

$$p(L) = 1 - e^{-WL/A} , \tag{I-4}$$

where W stands for the effective search width of the detection law and L represents the total length of the searcher's path within A. Let us emphasize, however, that while this development does indeed eliminate the difficulty of integrating $\gamma(t)$ along the relative track of the target, it only does so at the expense of sacrificing the realistic and often crucial concept of correlation along the path of the searcher.

2. THE KOOPMAN APPROACH MODIFIED

The Koopman framework has proved quite successful in a large variety of naval warfare problems, and has served as a guideline to many, more sophisticated later treatments. It is within the nature of our stated purpose, however, to investigate the shortcomings of

the formulation rather than emphasize its virtues. Correspondingly, we shall concentrate in the following upon the major limitations of the Koopman framework.

As will be recalled, the Koopman theory takes the a posteriori view of the search and detection problem whereby both target and searcher trajectories are assumed known. More often than not, however, such is not the case. In fact, the path of a military unit is well represented by a juxtaposition of deterministic pieces of path, the process of passing from one to any other in time being a random one to be sampled from some given distribution dependent on both tactical and physical parameters. The question to be answered, then, is one concerning the manner in which the probabilistic description inherent to the arrival process is to be incorporated in the Koopman equation. While we do recognize that an an attempt at addressing this question exists in the Koopman theory, it is important to note that limitations in the mathematical formulation thereof do not allow the preservation of correlations along the path of the searcher.

It is also worth noticing that the two parts of the game as envisioned in the Koopman formalism are not really independent of each other, in that looking can produce feedback on the trajectories of both players. In this sense, the framework is void of dynamics.

The formulation that we propose and develop to some extent in this work is designed to hopefully handle each and all of the points raised against the Koopman framework. It essentially consists in determining various physical quantities relevant to the search and detection game for any of a family of simple target paths, each one of which is assigned a given probability of having been chosen from the family. The actual value of the physical quantity of interest is then obtained by averaging over the family parameters.

The main conceptual virtue of this framework is that it is more responsive to the actual restrictions that reality imposes upon the search and detection game. This added responsiveness gets reflected not only in the possibility of choosing the path probabilities to suit the real case at hand, but also in the freedom of choosing the family in such a manner that it be indicative of the operational features of the actual setup while still simple enough to allow, in principal at least, for tractable solutions to equation (I-3). It is interesting to note that, unlike the Koopman formulation, all of these conceptual advantages are available without the loss of correlation along the path of the searcher. In fact, as we shall discuss later on in some detail, most of the relevant correlations are properly taken into account by the theory. Statements concerning the less conceptual advantages the framework offers are relegated to conclusions when the reader shall be more familiar with the language.

3. OUTLINE

For reasons of simplicity we have chosen to break the presentation to follow up into two parts. In the first chapter we shall address the relatively independent question of determining realistic probabilities of arrival, both because they are relevant in themselves and because, as it turns out, they are the necessary building blocks for calculating

probabilities of detection. The second chapter of this paper will show how nonuniform distributions of target arrival can be incorporated into the detection problem. Since we succeed in doing so without breaking up the searcher path in independent pieces, correlations along the path of the searcher are fully preserved in the formalism.

PART I. NONUNIFORM PROBABILITIES OF ARRIVAL

A. THE GENERAL FRAMEWORK

We pursue the program outlined in the Introduction to this paper by addressing first the relatively independent question of calculating realistic probability densities of target arrival. We thus let,

$$F(\vec{r},\omega) = 0 \tag{A-1}$$

be an arbitrary family of curves parameterized by the set ω and define the probability $a(\omega)$ that the target has chosen to approach the searching area along some path ω. Naturally,

$$\int_F d\omega \; a(\omega) = 1 \quad . \tag{A-2}$$

Now, if $G_\omega(\vec{\rho},t\,|\,\vec{\rho}_0,t_0)$ represents the probability density of arriving at $(\vec{\rho},t)$ from $(\vec{\rho}_0,t_0)$ along path ω, we write for the probability density of arrival at $(\vec{\rho},t)$ having started at $(\vec{\rho}_0,t_0)$,

$$G(\vec{\rho},t\,|\,\vec{\rho}_0,t_0) \equiv \int d\omega \; a(\omega)\, G_\omega(\vec{\rho},t\,|\,\vec{\rho}_0,t_0) \quad . \tag{A-3}$$

To proceed, we now specify

$$G_\omega(\vec{\rho},t\,|\,\vec{\rho}_0,t_0) = \delta\left(\vec{r}(\omega,t)-\vec{\rho}\right) \tag{A-4}$$

where $\vec{r}(\omega,t)$ describes the position of the moving target along the path ω. Correspondingly,

$$G(\vec{\rho},t\,|\,\vec{\rho}_0,t_0) = \int d\omega \; a(\omega) \; \delta\left(\vec{r}(\omega,t)-\vec{\rho}\right) \quad . \tag{A-5}$$

In fact,

$$\langle\vec{r}\rangle_t \equiv \int d\omega \; a(\omega) \; \vec{r}(\omega,t) = \int d\omega \; a(\omega) \int d\vec{\rho} \; \vec{\rho} \; \delta\left(\vec{r}(\omega,t)-\vec{\rho}\right)$$

$$= \int d\vec{\rho} \; \vec{\rho} \int d\omega \; a(\omega) \; \delta\left(\vec{r}(\omega,t)-\vec{\rho}\right) \; ,$$

which identifies $G(\vec{\rho}, t \mid \vec{\rho}_0, t_0)$ given in equation (A-5). For the reader who is not impressed with the rigor of our derivation, we note that one can easily continue the argument to show that (A-5) also has the appropriate higher moment.

Using the properties of the Dirac delta function, one immediately shows that $G(\vec{\rho}, t \mid \vec{\rho}_0, t_0)$ is properly normalized at all times if $a(\omega)$ satisfies equation (A-2). Indeed, using equation (A-5), we have:

$$\int d\vec{\rho} \; G(\vec{\rho}, t \mid \vec{\rho}_0, t_0) = \int d\omega \; a(\omega)$$

and hence,

$$\int d\vec{\rho} \; G(\vec{\rho}, t \mid \vec{\rho}_0, t_0) = 1 \quad .$$

It is noteworthy that the global correlation expressed in the requirement that the target reach $(\vec{\rho}, t)$ having started at $(\vec{\rho}_0, t_0)$ is here automatically translated into purely geometrical restrictions to be imposed upon the family,

$$\vec{r}(\omega, t_0) = \vec{\rho}_0$$

as well as upon the integration in equation (A-5). This interesting feature of the formalism shall prove of some importance later on in that it offers a convenient way of including correlations into the framework.

B. THE CASE OF THE ISOTOPIC WALK

To start the framework off, a sensible choice for the family of paths is made. For purposes of illustration and because they are indicative of many realistic cases, we analyze the family of all random walks starting at the origin and parameterized by

$$\omega \equiv \left[n, \left(\theta_i ; v_i ; \tau_i\right)_{i=1,2,\ldots,n} \right] . \tag{B-1}$$

Here, as displayed in figure 1, the set (θ_i, v_i, τ_i) stands for the angle, speed along, and time interval of the ith step in the walk, while n represents the number of steps considered. Specifically, for a random walk of n steps the family equation reads:

$$x_n(t) = \sum_{i=1}^{n} \alpha_i \cos \theta_i \quad ; \quad y_n(t) = \sum_{i=1}^{n} \alpha_i \sin \theta_i \tag{B-2}$$

$$\alpha_i \equiv v_i \tau_i \quad ; \quad t = \sum_{i=1}^{n} \tau_i \quad . \tag{B-3}$$

In this section we specialize t the simple case

$$\alpha_i = \alpha \quad ; \quad v_i = v \quad ; \quad i = 1, 2, \ldots, n \tag{B-4}$$

and correspondingly measure time in units of $\tau = \alpha / v$,

$$t = n\tau \quad . \tag{B-5}$$

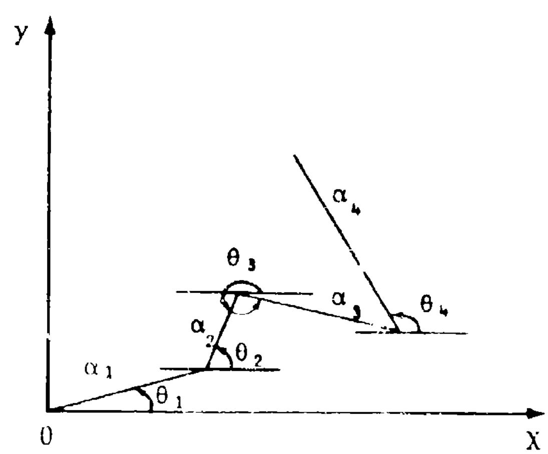

FIG. 1: THE FAMILY OF RANDOM WALKS

Therefore, the associated quantity $a(\omega)$ becomes the joint probability density $a(\theta_1, \theta_2, \ldots, \theta_n)$ that the family parameters take on the set of values $(\theta_1, \theta_2, \ldots, \theta_n)$ respectively, and

$$G(\vec{\rho},t|0,0) = \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \cdots \int_0^{2\pi} d\theta_n \, a(\theta_1,\theta_2,\ldots,\theta_n)_* \tag{B-6}$$

$$\delta\left(\sum_{i=1}^{n} \alpha \cos \theta_i - \sigma\right) \delta\left(\sum_{i=1}^{n} \alpha \sin \theta_i - \rho\right)$$

for (σ, ρ) the cartesian components of $\vec{\rho}$.

We now make the simplifying assumption that the decision of choosing any particular value for some given θ_i is independent of the past,

$$a(\theta_1,\theta_2,\ldots,\theta_n) = a_1(\theta_1)a_2(\theta_2) \cdots a_n(\theta_n) \quad . \tag{B-7}$$

Because (B-7) is tantamount to assuming that the probability distribution for any given angle θ_i is independent of the point in the ocean at which the walking unit finds itself, we shall refer to (B-7) as the "isotopic walk case." We hasten to add, however, that while of some methodological importance here, the isotopic walk assumption is by no means necessary. In fact, realistic correlations along the path of the walk will be properly taken into account in a later chapter of this paper.

Finally, to make matters as simple as possible, we specify:

$$a_i(\theta_i) = 1/2\pi \quad ; \qquad i=1,2,\ldots,n \tag{B-8}$$

while mentioning that the minor restriction imposed by (B-8) can be lifted with no difficulty at all.

Hence,

$$G(\vec{\rho},t|\vec{0},0) = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \cdots \int_0^{2\pi} d\theta_n \; * \tag{B-9}$$

$$\delta\left(\sum_{i=1}^{n} \alpha \cos \theta_i - \sigma\right) \delta\left(\sum_{i=1}^{n} \alpha \sin \theta_i - \rho\right) \quad .$$

To cast equation (B-9) into a more manageable form, we make use of the spectral representation for the Dirac delta function,

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; e^{ikx} \tag{B-10}$$

and have:

$$G(\vec{\rho},t|\vec{0},0) = \left(\frac{1}{2\pi}\right)^{n+2} \int_{-\infty}^{\infty} d\xi e^{-i\xi\sigma} \int_{-\infty}^{\infty} d\eta e^{-i\eta\rho} \quad * \tag{B-11}$$

$$\left[\int_0^{2\pi} d\theta e^{i\xi\alpha\cos\theta} e^{i\eta\alpha\sin\theta}\right]^n$$

Recognizing that

$$\int_0^{2\pi} d\theta e^{i\xi\alpha\cos\theta} \; e^{i\eta\alpha\sin\theta} = 2\pi\, J_0\left(\alpha\sqrt{\xi^2 + \eta^2}\right) \tag{B-12}$$

where $J_0\left(\alpha\sqrt{\xi^2 + \eta^2}\right)$ denotes the zeroth Bessel function of the corresponding argument,

$$G(\vec{\rho},t|\vec{0},0) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\xi e^{-i\xi\sigma} \int_{-\infty}^{\infty} d\eta e^{-i\eta\rho} \; J_0^{\,n}\left(\alpha\sqrt{\xi^2 + \eta^2}\right) . \tag{B-13}$$

We shall find it convenient to pass to polar coordinates in (ξ,η) ,

$$\xi = x\cos\phi \quad ; \quad \eta = x\sin\phi$$

and explicitly perform the angular integration,

$$G(\vec{\rho},t|\vec{0},0) = \frac{1}{2\pi} \int_0^{\infty} dx \; x J_0\left(x\sqrt{\sigma^2 + \rho^2}\right) J_0^{\,n}(\alpha x) \quad . \tag{B-14}$$

An immediate offshot of (B-14) is that

$$G(\vec{\rho}, t \mid \vec{0}, 0) = 0 \qquad \text{if} \qquad \sqrt{\sigma^2 + \rho^2} > n\alpha \tag{B-15}$$

as expected.

We can now start the simple but lengthy job of calculating $G(\vec{\rho}, t \mid \vec{0}, 0)$ for various values of n. Thus, for n=2, (see figure (2),

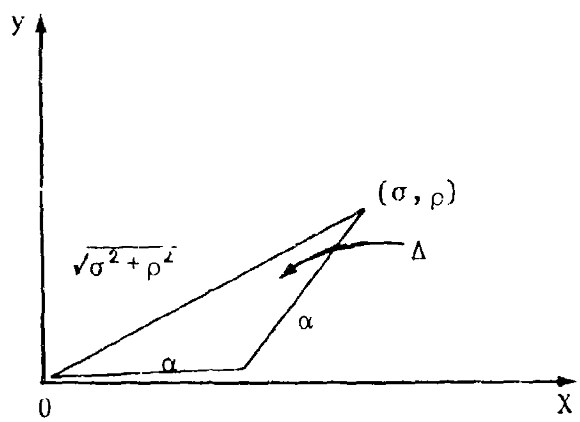

FIG. 2: GEOMETRY CONCERNING THE ARRIVAL PROBABILITY FOR n=2

$$G(\vec{\rho}, 2\tau \mid \vec{0}, 0) = \frac{1}{4\pi^2 \Delta} \tag{B-16}$$

where Δ represents the area under the triangle formed by the triplet $\left(\sqrt{\sigma^2 + \rho^2}; \alpha; \alpha\right)$. Naturally, in accordance with equation (B-15), the probability density of arrival for n=2 vanishes whenever the triplet fails to form a triangle. We also note that for the marginal case of measure zero,

$$\sqrt{\sigma^2 + \rho^2} = 2\alpha$$

where $\Delta = 0$, the probability density of arrival becomes infinite.

To handle the case $n > 2$, we could make repeated use of the equality

$$\int_0^{\pi} d\theta \; J_0\left(\sqrt{Z^2 + \zeta^2 - 2Z\zeta \cos\theta}\right) = J_0(Z) J_0(\zeta) \tag{B-17}$$

to reduce the power of the Bessel function to $n=2$, and then employ (B-16), but restrain from doing so here. Rather, we develop in the next section a more general technique of handling the calculational problem that shall provide answers at a much smaller expense, and at the same time, indicate the manner in which generalizations to the simple example considered here are to be approached.

C. THE EQUIVALENT ITERATIVE FORMULATION

To begin with, recall the result (B-13) from the previous section,

$$G(\vec{\rho}, n\tau \,|\, 0,0) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\xi e^{-i\xi\sigma} \int_{-\infty}^{\infty} d\eta e^{-i\eta\rho} \, J_0^{\,n}\left(\alpha\sqrt{\xi^2+\eta^2}\right), \tag{C-1}$$

but avoid explicit integration on (ξ, η). Instead, make use of the convolution theorem for the Fourier transform,

$$\int_{-\infty}^{\infty} d\mathbf{x} e^{-ik\mathbf{x}} f(\mathbf{x}) g(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\ell \; \hat{f}(\ell) \hat{g}(k-\ell)$$

where

$$\hat{f}(\ell) \equiv \int_{-\infty}^{\infty} d\mathbf{x} e^{-i\ell} f(\mathbf{x})$$

and

$$\hat{g}(\ell) \equiv \int_{-\infty}^{\infty} d\mathbf{x} e^{-i\ell\mathbf{x}} g(\mathbf{x})$$

onto the product $J_0\left(\alpha\sqrt{\xi^2+\eta^2}\right) J_0^{\,n-1}\left(\alpha\sqrt{\xi^2+\eta^2}\right)$ to have:

$$G(\vec{\rho},n\tau|\vec{0},0) = \frac{2}{(2\pi)^3}\int_{-\infty}^{\infty} d\xi e^{-i\xi\sigma}\int_{-\infty}^{\infty} dy\, \frac{\theta[\alpha^2-(\rho-y)^2]}{\sqrt{\alpha^2-(\rho-y)^2}} \quad *$$

$$\cos\left(\xi\sqrt{\alpha^2-(\rho-y)^2}\right)\int_{-\infty}^{\infty} d\nu e^{-i\nu y}\, J_0^{\,n-1}\left(\alpha\sqrt{\xi^2+\nu^2}\right)$$

and continuing,

$$G(\vec{\rho},n\tau|\vec{0},0) = \frac{2}{(2\pi)^3}\int_{-\infty}^{\infty} dy\, \frac{\theta[\alpha^2-(\rho-y)^2]}{2\sqrt{\alpha^2-(\rho-y)^2}}\int_{-\infty}^{\infty} d\nu e^{-i\nu y} \quad *$$

$$\int_{-\infty}^{\infty} d\mathbf{x}\left[\delta\left(\sigma-\mathbf{x}-\sqrt{\alpha^2-(\rho-y)^2}\right)+\delta\left(\sigma-\mathbf{x}+\sqrt{\alpha^2-(\rho-y)^2}\right)\right] \quad *$$

$$\int_{-\infty}^{\infty} d\mu e^{-i\mu\mathbf{x}}\, J_0^{\,n-1}\left(\alpha\sqrt{\mu^2+\nu^2}\right).$$

Hence, recognizing that

$$\delta\left[(\sigma-\mathbf{x})^2+(\rho-y)^2-\alpha^2\right] = \frac{1}{2\sqrt{\alpha^2-(\rho-y)^2}}\left[\delta\left(\sigma-\mathbf{x}-\sqrt{\alpha^2-(\rho-y)^2}\right)\right.$$

$$\left.+\;\delta\left(\sigma-\mathbf{x}+\sqrt{\alpha^2-(\rho-y)^2}\right)\right]$$

and correspondingly dropping the step function $\theta[\alpha^2 - (\rho - y)^2]$ whose value is always one on the circle

$$(\sigma - x)^2 + (\rho - y)^2 = \alpha^2 \quad ,$$

the probability density of arrival at $\vec{\rho}$ in n steps becomes:

$$G(\vec{\rho}, n\tau | \vec{0}, 0) = \frac{2}{(2\pi)^3} \int_{-\infty}^{\infty} dx\,dy\ \delta\left[(\sigma - x)^2 + (\rho - y)^2 - \alpha^2\right] *$$

$$\int_{-\infty}^{\infty} d\mu e^{-i\mu x} \int_{-\infty}^{\infty} d\nu e^{-i\nu y}\ J_0^{n-1}\left(\alpha\sqrt{\mu^2 + \nu^2}\right) .$$

Finally, since in virtue of equation (C-1),

$$\int_{-\infty}^{\infty} d\mu e^{-i\mu X} \int_{-\infty}^{\infty} d\nu e^{-i\nu y}\ J_0^{n-1}\left(\alpha\sqrt{\mu^2 + \nu^2}\right)$$

$$= (2\pi)^2\ G\left(\vec{r}, (n-1)\tau | \vec{0}, 0\right),$$

for $\vec{r}$ the two-dimensional vector of components (x, y), our main result reads:

$$\boxed{G(\vec{\rho}, n\tau | \vec{0}, 0) = \int d\vec{r}\ \frac{1}{\pi}\ \delta\left[(\vec{\rho} - \vec{r})^2 - \alpha^2\right] G\left(\vec{r}, (n-1)\tau | \vec{0}, 0\right)} \qquad \text{(C-3)}$$

A quick glance at (C-3) will now convince the reader that the physical content embodied therein is surprisingly simple: the probability $G(\vec{\rho}, n\tau | \vec{0}, 0) d\vec{\rho}$ of arriving at $\vec{\rho}$ in n steps having started at the origin is given by the probability $G\left(\vec{r}, (n-1)\tau | \vec{0}, 0\right) d\vec{r}$ of arriving at $\vec{r}$ in (n-1) steps having started at the origin folded into the probability $\frac{1}{\pi}\ \delta[(\vec{\rho} - \vec{r})^2 - \alpha^2]\ d\vec{\rho}$ of taking the last step from $\vec{r}$ to $\vec{\rho}$ and integrated over all intermediate points $\vec{r}$.

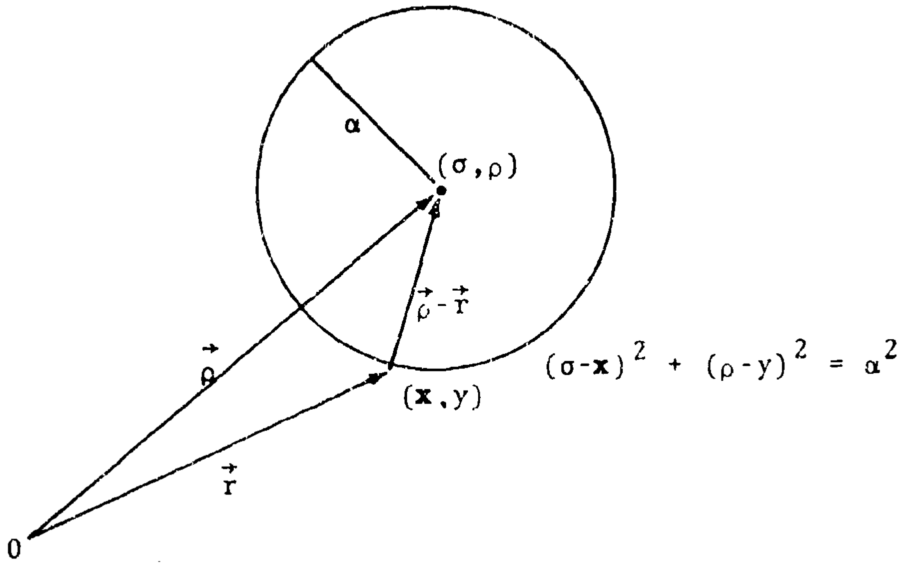

FIG. 3: GEOMETRY OF THE ITERATIVE SOLUTION FOR AN UNIFORM ISOTOPIC WALK

Also note, that since the one step probability identified above is properly normalized,

$$\int d\vec{\rho}\ \frac{1}{\pi}\ \delta[(\vec{\rho}-\vec{r})^2-\alpha^2] = 1 \quad , \tag{C-4}$$

as one can easily ascertain by explicit integration, the normalization condition

$$\int d\vec{r}\ G\left(\vec{r},(n-1)\tau\,|\,\vec{0},0\right) = 1 \tag{C-5}$$

implies via (C-3),

$$\int d\vec{\rho}\ G(\vec{\rho},n\tau\,|\,\vec{0},0) = 1$$

as expected.

The very reasonable physical interpretation that we have just given equation (C-3), suggests the possibility of guessing at the results to be expected for cases that are somewhat more complex than the simple uniform isotopic case analyzed so far. Thus, to lift the restriction of equation (B-8), we allow $a_i(\theta_i)$ to be any arbitrary, properly normalized set of functions and write for the probability density of arrival at $\vec{\rho}$ in n steps,

$$G(\vec{\rho},n\tau|\vec{0},0) = \int d\vec{r}\; 2a_n(\vec{\rho}-\vec{r})\;\delta[(\vec{\rho}-\vec{r})^2-\alpha^2]\; G\Big(\vec{r},(n-1)\tau|\vec{0},0\Big) \quad \text{(C-7)}$$

where (see figure 4):

$$a_n(\vec{\rho}-\vec{r}) \equiv a_n(\theta_n^0) \quad ; \quad \sin\theta_n^0 = \frac{(\rho-y)}{\alpha} \quad .$$

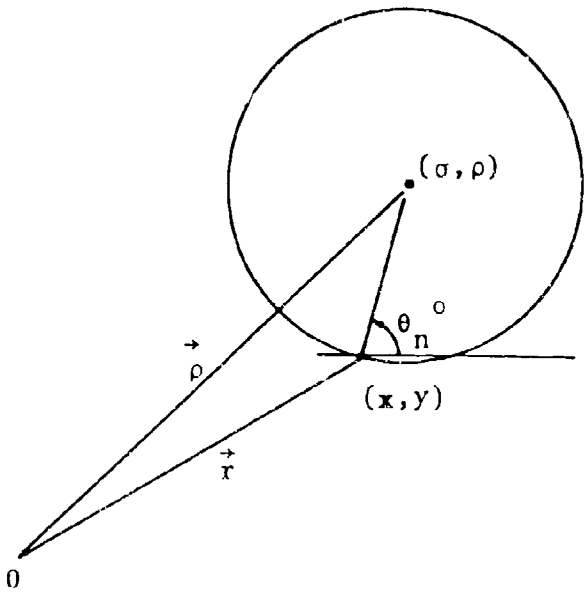

FIG. 4: GEOMETRY OF THE ITERATIVE SOLUTION FOR A NON-UNIFORM ISOTOPIC WALK

That (C-7) is, in fact, the correct answer can be checked, of course, with the technique that led to (C-3), but in view of the transparent physical content of (C-7) that is hardly warranted

Needless to say, the few cases mentioned above as illustrative instances do not exhaust the set of configurations that can be handled by the iterative technique, but we shall leave it to the patience of the interested reader the job of looking through them. The one configuration, however, that involves relaxing assumption (B-7) as well as the ensuing discussion are considered sufficiently delicate to warrant a separate later section. As to the manner in which actual numbers can be obtained from the rather formal results of this section, we shall have little to say at this time, but note that quite powerful Monte Carlo methods are

available for handling exactly this kind of iterative equations. Correspondingly, when needed, solutions for any value of n can be obtained at much less expense than experienced in the previous section.

D. THE CASE OF THE CORRELATED WALK

For all their value, the results that have been obtained so far suffer from the overall constraint embodied in (B-7). We shall now show that the main features of what understanding has been gained for the case of the isotopic walk survive the introduction of correlations.

Consider again,

$$\alpha_i = \alpha \quad ; \qquad v_i = v \quad ; \qquad i = 1, 2, \ldots, n$$

and make the following assumption of nonisotopy:

$$a(\theta_1, \theta_2, \ldots, \theta_n) = a_1(\theta_1) \prod_{i=2}^{n} a_i \left(\theta_i \,\middle|\, \sum_{j=1}^{i-1} \alpha \cos \theta_j ; \sum_{j=1}^{i-1} \alpha \sin \theta_j \right) \qquad \text{(D-1)}$$

to replace (B-7). Here

$$a_i \left(\theta_i \,\middle|\, \sum_{j=1}^{i-1} \alpha \cos \theta_j ; \sum_{j=1}^{i-1} \alpha \sin \theta_j \right)$$

represent the conditional probability distributions for sampling θ_i given that the walking unit is at the point in the ocean of coordinates,

$$x_i = \sum_{j=1}^{i-1} \alpha \cos \theta_j \quad ; \qquad y_i = \sum_{j=1}^{i-1} \alpha \sin \theta_j$$

along its path. Correspondingly,

$$G(\vec{\rho},n\tau|\vec{0},0) = \int d\theta_1 a_1(\theta_1) \int \prod_{i=2}^{n} d\theta_i \, *$$

$$a_i\left(\theta_i \,\middle|\, \sum_{j=1}^{i-1} \alpha \cos\theta_j ; \; \sum_{j=1}^{i-1} \alpha \sin\theta_j \right) * \tag{D-2}$$

$$\delta\left(\sum_{k=1}^{n} \alpha \cos\theta_k - \sigma\right) \delta\left(\sum_{k=1}^{n} \alpha \sin\theta_k - \rho\right)$$

or, rather,

$$G(\vec{\rho},n\tau|\vec{0},0) = \int \prod_{i=1}^{n} d\theta_i \, dx_i \, dy_i \; a_i \; (\theta_i | x_j, y_i) \; \delta\left(x_i - \sum_{j=1}^{i-1} \alpha \cos\theta_j\right) *$$

$$\text{(D-3)}$$

$$\delta\left(y_i - \sum_{j=1}^{i-1} \alpha \sin\theta_j\right) \delta\left(\sum_{k=1}^{n} \alpha \cos\theta_k - \sigma\right) \delta\left(\sum_{k=1}^{n} \alpha \sin\theta_k - \rho\right)$$

and using again the spectral representation of the Dirac delta function together with the convolution theorem for Fourier transforms, we obtain

$$\boxed{G(\vec{\rho},n\tau|\vec{0},0) = \int d\vec{r} \; 2a_n(\vec{\rho}-\vec{r}|\vec{r}) \; \delta[(\vec{\rho}-\vec{r})^2 - a^2] \; G\left(\vec{r},(n-1)\tau|\vec{0},0\right)}$$

where

$$a_n(\vec{\rho}-\vec{r}|\vec{r}) \equiv a_n\left(\theta_n^0 \,\middle|\, \vec{r}\right) \quad ; \qquad \sin\theta_n^0 = \frac{(\rho - y)}{\alpha}$$

It then follows that all conceptual as well as calculational advantages that go along with the iterative type solutions of section C, remain also valid for the correlated configuration considered here. It is perhaps worth noticing at this time that while (D-1) is not the most general correlation along the path that can be envisioned, it should account for the majority of realistic situations one might be concerned with. Correspondingly, we shall pursue the question of correlations no further.

E. THE CASE OF THE BOUNDED WALK

In many realistic cases of interest to us, intelligence information is available concerning the existence of a bounded region in the ocean within which the approaching enemy vessel is most likely to be found. Correspondingly, the study of the arrival probability density in the presence of given boundary conditions becomes relevant. Within our framework, the question of boundary conditions can be treated with relative ease in the same spirit the question of correlations along the path of the walk has been, i.e., by translating the physical restrictions imposed therein into geometrical constraints for (A-5).

To illustrate the basic concepts involved, consider the following simple boundary conditions,

$$0 \leq x\,,y \leq L \tag{E-1}$$

corresponding to a square shaped limitation upon the walk of the target. To ensure that (E-1) is respected, it shall suffice to limit the set of paths on which the integration in (A-5) is performed to include only those that independently satisfy (E-1). Therefore, if the case of the uniform isotopic walk with constant speed and step is chosen again,

$$G_{BC}(\vec{\rho},n\tau|\vec{0},0) = \left(\frac{1}{2\pi}\right)^n \int \prod_{i=1}^{n} d\theta_i \; \theta\left(L - \sum_{j=1}^{i} \alpha \cos \theta_j\right) *$$

$$\theta\left(\sum_{j=1}^{i} \alpha \cos \theta_j\right) \theta\left(\sum_{j=1}^{i} \alpha \sin \theta_j\right) \theta\left(L - \sum_{j=1}^{i} \alpha \sin \theta_j\right) * \tag{E-2}$$

$$\delta\left(\sum_{j=1}^{n} \alpha \cos \theta_j - \sigma\right) \delta\left(\sum_{j=1}^{n} \alpha \sin \theta_j - \rho\right) \quad .$$

We can now attempt to reduce (E-2) to an iterative solution, employing the techniques developed in Section C. In fact, using the spectral representation for the step function.

$$\theta(x) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} d\omega \, \frac{e^{i\omega x}}{\omega + i\varepsilon}$$

one can show that:

$$\boxed{\begin{aligned} G_{BC}(\vec{\rho}, n\tau|\vec{0},0) &= \theta(L-\sigma)\ \theta(\sigma)\ \theta(L-\rho)\ \theta(\rho) \ * \\ &\int d\vec{r}\, \frac{1}{\pi}\, \delta[(\vec{\rho}-\vec{r})^2 - \alpha^2]\ G_{BC}\Big(\vec{r},(n-1)\tau|\vec{0},0\Big) \end{aligned}} \tag{E-3}$$

which displays the same intuitively simple content that its predecessors have.

The attentive reader will have noticed the little swindle that has been perpetrated in the definition of $G_{BC}(\vec{\rho}, n\tau|\vec{0},0)$. Specifically, since,

$$K_n \equiv \int d\vec{\rho}\ G_{BC}(\vec{\rho}, n\tau|\vec{0},0) = \int d\omega\ a(\omega)$$

where the right hand side integral extends only over those ω paths that satisfy (E-1), the probability density is not normalized to unity. In fact, one can easily argue that

$$K_n < 1 \quad .$$

To compensate for the swindle, let us define,

$$K_n(\vec{r}_0) \equiv \int d\vec{\rho}\ G_{BC}(\vec{\rho}, n\tau|\vec{r}_0,0) \tag{E-4}$$

and use the symmetry property

$$G_{BC}(\vec{\rho}, n\tau|\vec{r}_0,0) = G_{BC}(\vec{r}_0, n\tau|\vec{\rho},0) \tag{E-5}$$

to write

$$K_n(\vec{r}_0) = \int d\vec{\rho}\ p_{BC}(\vec{r}_0, n\tau | \vec{\rho}, 0) \quad .$$

Hence, in virtue of (E-3)

$$K_0(\vec{r}_0) = 1$$

$$K_n(\vec{r}_0) = \theta(L-x_0)\ \theta(x_0)\ \theta(L-y_0)\ \theta(y_0) \quad * \tag{E-7}$$

$$\int d\vec{r}\ \frac{1}{\pi}\ \delta[(\vec{r}_0-\vec{r})^2-\alpha^2]\ K_{n-1}(\vec{r}) \quad ,$$

and for each value of n , the properly normalized probability density is obtained dividing through the appropriate solution of (E-3) by the corresponding solution of (E-7) taken at $\vec{r}_0=\vec{0}$,

$$\hat{G}_{BC}(\vec{\rho}, n\tau | \vec{0}, 0) = \frac{G_{BC}(\vec{\rho}, n\tau | \vec{0}, 0)}{K_n(\vec{0})} \tag{E-8}$$

PART II. DETECTION PROBABILITIES

Having modified the Koopman approach to account for non-uniform probabilities of arrival, we are now prepared to address in some detail the problem of target detection within the framework of a search and detection game. As the reader will recall, the main point of our proposal is to allow the target to arrive in the vicinity of the searcher along any one of a family of possible paths parameterized by the multidimensional set ω ,

$$F(\vec{r};\ \omega) = 0 \quad .$$

Each path within F is subsequently assigned a given probability $a(\omega)$ of having been chosen by the approaching target, and

$$\int_F d\omega\, a(\omega) = 1 \quad .$$

Hence, if $\bar{p}(\omega,\ t)$ represents the probability of no detection through time T for a target approaching the searcher along path ω , we write for the probability of no detection through time t ,

$$\bar{p}(t) = \int_F d\omega\, a(\omega)\bar{p}(\omega,\ t) \quad .$$

A. THE DETECTION PROBABILITY FOR A SPECIFIED SEARCH PATTERN

Let the searcher proceed along the specified search pattern,

$$\vec{r}_S = \vec{r}_S(t) \quad . \qquad \text{(A-1)}$$

Then, as indicated in Koopman's book, the probability of no detection through time t is obtained by solving the following differential equation:

$$d\bar{p}(\omega,t) = -\bar{p}(\omega,t)\ \gamma[\vec{r}_T(\omega,t) - \vec{r}_S(t)]\ dt \qquad \text{(A-2)}$$

subject to the initial condition

$$\bar{p}(\omega,0) = \bar{p}_0 \quad .$$

Here, and everywhere else, $\vec{r}_T(\omega,t)$ represents the position of the target along its trajectory at time t .

To proceed, we take $\bar{p}_0 = 1$, write equation (A-2) as,

$$\bar{p}(\omega,t) = 1 - \int_0^t d\tau \ \gamma[\vec{r}_T(\omega,\tau) - \vec{r}_S(\tau)] \ \bar{p}(\omega,\tau) \tag{A-3}$$

and Newman iterate (A-3) into the form,

$$\bar{p}(\omega,t) = 1 + \sum_{n=1}^{\infty} (-1)^n \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ \ldots \int_0^{\tau_{n-1}} d\tau_n \ * \tag{A-4}$$

$$T\left[\gamma\left(\vec{r}_T(\omega,\tau_1) - \vec{r}_S(\tau_1)\right) \gamma\left(\vec{r}_T(\omega,\tau_2) - \vec{r}_S(\tau_2)\right) \ldots \gamma\left(\vec{r}_T(\omega,\tau_n) - \vec{r}_S(\tau_n)\right)\right] .$$

In equation (A-4) we employ the symbolic notation T to represent the time ordering of the product of γ's within the square brackets, i.e.,

$$T \Rightarrow (0 \leq \tau_1 \leq \tau_2 \leq \ldots \leq \tau_n \leq t) \quad .$$

We make use of the symmetry of the time-ordered expression upon interchange of any of the coordinates $\tau_1, \tau_2, \ldots, \tau_n$, to symmetrize the integration interval with respect to the n indices. For $n = 2$ we recognize

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ T\left[\gamma\left(\vec{r}_T(\omega,\tau_1) - \vec{r}_S(\tau_1)\right) \gamma\left(\vec{r}_T(\omega,\tau_2) - \vec{r}_S(\tau_2)\right)\right] =$$

$$\int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 \ T\left[\gamma\left(\vec{r}_T(\omega,\tau_1) - \vec{r}_S(\tau_1)\right) \gamma\left(\vec{r}_T(\omega,\tau_2) - \vec{r}_S(\tau_2)\right)\right] =$$

$$\frac{1}{2} \int_0^t d\tau_1 \int_0^t d\tau_2 \ T\left[\gamma\left(\vec{r}_T(\omega,\tau_1) - \vec{r}_S(\tau_1)\right) \gamma\left(\vec{r}_T(\omega,\tau_2) - \vec{r}_S(\tau_2)\right)\right] \quad .$$

For general n we can similarly carry out the n! permutations of the n indices and extend the integration region over the n-dimensional cube from zero to t. Each of the n! permuted time-ordered regions contributes equally, and we therefore write

$$\bar{p}(\omega,t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^t d\tau_1 \int_0^t d\tau_2 \cdots \int_0^t d\tau_n \ *$$

$$T\left[\gamma\left(\vec{r}_T(\omega,\tau_1)-\vec{r}_S(\tau_1)\right) \gamma\left(\vec{r}_T(\omega,\tau_2)-\vec{r}_S(\tau_2)\right) \cdots \gamma\left(\vec{r}_T(\omega,\tau_n)-\vec{r}_S(\tau_n)\right)\right] .$$

Next, we employ the identity

$$\gamma\left(\vec{r}_T(\omega,\tau_i)-\vec{r}_S(\tau_i)\right) = \int d\vec{r}_i \ \gamma\left(\vec{r}_i-\vec{r}_S(\tau_i)\right) \delta\left(\vec{r}_T(\omega,\tau_i)-\vec{r}_i\right)$$

to write for the probability of no detection,

$$\bar{p}(\omega,t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^t d\tau_1 \, d\vec{r}_1 \int_0^t d\tau_2 d\vec{r}_2 \cdots \int_0^t d\tau_n d\vec{r}_n \ * \tag{A-5}$$

$$T\left[\gamma\left(\vec{r}_1-\vec{r}_S(\tau_1)\right) \delta\left(\vec{r}_T(\omega,\tau_1)-\vec{r}_1\right) \cdots \gamma\left(\vec{r}_n-\vec{r}_S(\tau_n)\right) \delta\left(\vec{r}_T(\omega,\tau_n)-\vec{r}_n\right)\right] .$$

Hence, the actual probability of no detection,

$$\bar{p}(t) = \int d\omega \ a(\omega) \ \bar{p}(\omega,t)$$

becomes:

$$\bar{p}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^t d\tau_1\, d\vec{r}_1 \int_0^t d\tau_2\, d\vec{r}_2 \cdots \int_0^t d\tau_n\, d\vec{r}_n \; *$$

$$T\Big[\gamma\big(\vec{r}_1-\vec{r}_s(\tau_1)\big)\,\gamma\big(\vec{r}_2-\vec{r}_s(\tau_2)\big)\cdots\gamma\big(\vec{r}_n-\vec{r}_s(\tau_n)\big) \; * \qquad \text{(A-6)}$$

$$\int d\omega\; a(\omega)\; \delta\big(\vec{r}_T(\omega,\tau_1)-\vec{r}_1\big)\,\delta\big(\vec{r}_T(\omega,\tau_2)-\vec{r}_2\big)\cdots\delta\big(\vec{r}_T(\omega,\tau_n)-\vec{r}_n\big)\Big] \;,$$

where, of course, in obtaining (A-6) we have had to assume that the infinite sum over n in equation (A-5) has the necessary convergence properties to justify our piecewise application of the integral operator $\int d\omega\; a(\omega)$.

In equation (A-6), the ω integration extends only over the family of solutions to the simultaneous set of equations,

$$\vec{r}_T(\omega,\tau_i) = \vec{r}_i \quad (i = 1,2,\ldots,n)$$

as ensured by the Dirac delta functions appearing there. Within this family, however, the probability corresponding to any given path is expressable as the product,

$$a(\omega) = \prod_{i=1}^{n} a_i(\omega)$$

where a_i represents the probability of selecting from all trajectories connecting $(\vec{r}_{i-1},\tau_{i-1})$ to $(\vec{r}_i,\ \tau_i)$ the particular one that coincides with the segment of the ω-path that is contained therein.

If the Markovian type assumption is made, whereby $a_i(\omega)$ depend only on the set ω_i parameterizing the family of curves connecting $(\vec{r}_{i-1}, \tau_{i-1})$ to $(\vec{r}_i, \tau_i)$, it can be easily shown that,

$$\int d\omega\ a(\omega)\ \delta\left(\vec{r}_T(\omega,\tau_1)-\vec{r}_1\right)\delta\left(\vec{r}_T(\omega,\tau_2)-\vec{r}_2\right)\ \ldots\ \delta\left(\vec{r}_T(\omega,\tau_n)-\vec{r}_n\right)$$

$$= \prod_{i=1}^{n} G\left(\vec{r}_i, \tau_i \,\middle|\, \vec{r}_{i-1}, \tau_{i-1}\right)$$

for $\vec{r}_0$ the initial target position, $\tau_0 = 0$, and where $G\left(\vec{r}_i, \tau_i \,\middle|\, \vec{r}_{i-1}, \tau_{i-1}\right)$ are the probability densities of arrival discussed in Part I of this paper.

Consequently,

$$\bar{p}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^t \prod_{i=1}^{n} d\tau_i d\vec{r}_i\ \gamma\left(\vec{r}_i \ldots \vec{r}_s(\tau_i)\right)\ G\left(\vec{r}_i,\tau_i \middle| \vec{r}_{i-1},\tau_{i-1}\right) . \qquad \text{(A-7)}$$

It is perhaps worth noticing that the disappearance of the symbolic time ordered product in equation (A-7) is only formal in that each $G\left(\vec{r}_i,\tau_i \middle| \vec{r}_{i-1},\tau_{i-1}\right)$ implicitly carries with it a causal step function, $\theta\left(\tau_i - \tau_{i-1}\right)$ ensuring that all propagation takes place forward in time.

One way to render equation (A-7) useful, would be to define:

$$\psi_n\left(\vec{r}_{n+1},\tau_{n+1} \middle| \vec{r}_0,\tau_0\right) \equiv \int_0^t d\tau_1 d\vec{r}_1 \ldots \int_0^t d\tau_n d\vec{r}_n \quad * \qquad \text{(A-8)}$$

$$G\left(\vec{r}_{n+1},\tau_{n+1} \middle| r_0,\tau_0\right) \prod_{i=1}^{n} \gamma\left(\vec{r}_i - \vec{r}_s(\tau_i)\right) G\left(\vec{r}_i,\tau_i \middle| \vec{r}_{i-1},\tau_{i-1}\right) ,$$

and correspondingly write for the probability of no detection,

$$\bar{p}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^t d\tau_n d\vec{r}_n \; \gamma\left(\vec{r}_n - \vec{r}_s(\tau_n)\right) \; \psi_{n-1}\left(\vec{r}_n, \tau_n \middle| \vec{r}_0, \tau_0\right) \tag{A-9}$$

Although not necessarily transparent, there is virtue in having written $\bar{p}(t)$ in the form of equation (A-9). The advantage is contained in the observation that ψ_n thus defined satisfy an iterative type equation of the form

$$\psi_n\left(\vec{r}_{n+1}, \tau_{n+1} \middle| \vec{r}_0, \tau_0\right) = \int_0^{\tau_{n+1}} d\tau_n d\vec{r}_n \; G\left(\vec{r}_{n+1}, \tau_{n+1} \middle| \vec{r}_n, \tau_n\right) \; * \tag{A-10}$$

$$\gamma\left(\vec{r}_n - \vec{r}_s(\tau_n)\right) \; \psi_{n-1}\left(\vec{r}_n, \tau_n \middle| \vec{r}_0, \tau_0\right)$$

with,

$$\psi_0\left(\vec{r}_1, \tau_1 \middle| \vec{r}_0, \tau_0\right) \equiv G\left(\vec{r}_1, \tau_1 \middle| \vec{r}_0, \tau_0\right) . \tag{A-10'}$$

In fact, if we choose to identify $\psi_n\left(\vec{r}_{n+1}, \tau_{n+1} \middle| \vec{r}_0, \tau_0\right)$ with the probability of target arrival at $\left(\vec{r}_{n+1}, \tau_{n+1}\right)$ in the presence of an n-uple attempt at detection, equation (A-10) obtains the following simple physical interpretation (see figure 5): the probability of arrival at $\left(\vec{r}_{n+1}, \tau_{n+1}\right)$ in the presence of an n-uple attempt at detection, $\psi_n\left(\vec{r}_{n+1}, \tau_{n+1} \middle| \vec{r}_0, \tau_0\right) d\vec{r}_{n+1}$, is obtained by multiplying the probability of arrival at some point $\left(\vec{r}_n, \tau_n\right)$ in the presence of an (n-1)-uple attempt at detection, $\psi_{n-1}\left(\vec{r}_n, \tau_n \middle| \vec{r}_0, \tau_0\right) d\vec{r}_n$, with the instantaneous probability of detection $\gamma\left(\vec{r}_n - \vec{r}_s(\tau_n)\right) d\tau_n$ by one glimpse at τ_n, propagating the product thus described from $\left(\vec{r}_n, \tau_n\right)$ to $\left(\vec{r}_{n+1}, \tau_{n+1}\right)$ via the "propagator" $G\left(\vec{r}_{n+1}, \tau_{n+1} \middle| \vec{r}_n, \tau_n\right) d\vec{r}_{n+1}$ and then integrating over all the allowed intermediary points $\left(\vec{r}_n, \tau_n\right)$.

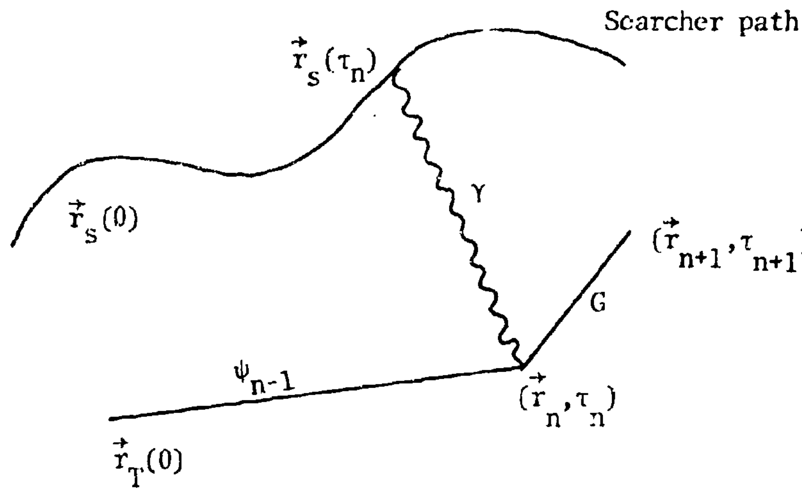

FIG. 5: GEOMETRY OF THE ITERATIVE SOLUTION FOR DETECTION ALONG A SPECIFIED SEARCH PATTERN

It is quite important to recognize the crucial role that the propagator plays in the formalism discussed above; it is precisely the presence of $G\left(\vec{r}_{n+1},\tau_{n+1}\middle|\vec{r}_n,\tau_n\right)$ in equation (A-10) that properly accounts for whatever correlations there exist along the path of the searcher, in that the propagator is responsible for the target arriving at the sight of the (n+1)th attempt at detection with full memory of the point $\left(\vec{r}_n,\tau_n\right)$ where the nth attempt at detection took place. In fact, if one were to arbitrarily break the correlation by replacing $G\left(\vec{r}_{n+1},\tau_{n+1}\middle|\vec{r}_n,\tau_n\right)$ in equation (A-10) with $G\left(\vec{r}_{n+1},\tau_{n+1}\middle|\vec{r}_0,\tau_0\right)$ the entire framework would automatically collapse into a Koopman-type result for $\bar{p}(t)$. Indeed, upon defining

$$\Phi_{n-1}(t) \equiv \int_0^t d\tau_n d\vec{r}_n\ \gamma\left(\vec{r}_n-\vec{r}_s(\tau_n)\right)\ \psi_{n-1}\left(\vec{r}_n,\tau_n\middle|\vec{r}_0,\tau_0\right), \tag{A-11}$$

and performing the indicated replacement, equation (A-10) becomes:

$$\Phi_n(t) = \Phi_{n-1}(t) \int_0^t d\tau d\vec{r}\ \gamma\left(\vec{r}-\vec{r}_s(\tau)\right)\ G\left(\vec{r},\tau\middle|\vec{r}_0,\tau_0\right) \tag{A-12}$$

and correspondingly,

$$\bar{p}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \Phi_{n-1}(t) \quad . \tag{A-13}$$

But, in virtue of equation (A-12)

$$\Phi_n(t) = \left[\int_0^t d\tau d\vec{r} \; \gamma\left(\vec{r}-\vec{r}_s(\tau)\right) \; G\left(\vec{r},\tau \middle| \vec{r}_0,\tau_0\right) \right]^{n+1} \tag{A-14}$$

so that

$$\bar{p}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left[\int_0^t d\tau d\vec{r} \; \gamma\left(\vec{r}-\vec{r}_s(\tau)\right) \; G\left(\vec{r},\tau \middle| \vec{r}_0,\tau_0\right) \right]^{n} \tag{A-15}$$

or, rather,

$$\bar{p}(t) = e^{-\int_0^t d\tau d\vec{r} \; \gamma\left(\vec{r}-\vec{r}_s(\tau)\right) \; G\left(\vec{r},\tau \middle| \vec{r}_0,\tau_0\right)} \tag{A-16}$$

to be compared with equation (I-3). If one is further willing to assume that $G\left(\vec{r},\tau \middle| \vec{r}_0,\tau_0\right)$ represents uniform arrival within an ocean region of area A, equation (A-16) simply reduces to (equation (I-4)) as expected.

Returning to the general formalism as comprised of equations (A-9) and (A-10), we finally note that it displays the appropriate structure to render the Monte Carlo numerical methods mentioned in Part I equally useful to the detection problem studied here.

B. GLOBAL DETECTION PROBABILITIES

In the foregoing discussion, the path of the searcher has been arbitrarily specified and the principle of biasing the set of possible paths in accordance with $a(\omega)$ used only in the description of target motion. The distinct advantage of doing so lies in the possibility of using the formalism thus developed for analyzing search pattern optimization problems. Sometimes, however, there is value in calculating global probabilities of detection, whereby both the searcher and the target motion are treated via the biasing principle. When such is the case, we shall use

$$d\bar{p}(\sigma,\omega;t) = - \bar{p}(\sigma,\omega;t) \; \gamma[\vec{r}_T(\omega,t) - \vec{r}_S(\sigma,t)] \; dt \tag{B-1}$$

to replace equation (A-2), where σ and ω represent the family of trajectory parameters for the searcher and target, respectively. Therefore,

$$\bar{p}(\sigma,\omega;t) = 1 - \int_0^t d\tau \ \gamma[\vec{r}_T(\omega,\tau) - \vec{r}_S(\sigma,\tau)]\bar{p}(\sigma,\omega;\tau) \tag{B-2}$$

and Newman iterating (B-2),

$$\bar{p}(\sigma,\omega;t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^t d\tau_1 \ \ldots \int_0^t d\tau_n \ * \tag{B-3}$$

$$T\left[\gamma\left(\vec{r}_T(\omega,\tau_1) - \vec{r}_S(\sigma,\tau_1)\right) \ldots \gamma\left(\vec{r}_T(\omega,\tau_n) - \vec{r}_S(\sigma,\tau_n)\right)\right] \ .$$

Just as before, we now use

$$\gamma\left(\vec{r}_T(\omega,\tau_i) - \vec{r}_S(\sigma,\tau_i)\right) =$$

$$\int d\vec{r}_i d\rho_i \ \gamma(\vec{r}_i - \rho_i) \ \delta\left(\vec{r}_T(\omega,\tau_i) - \vec{r}_i\right) \ \delta\left(\vec{r}_S(\sigma,\tau_i) - \rho_i\right)$$

to write equation (B-3) in the form:

$$\bar{p}(\sigma,\omega;t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^t (d\tau_1 d\vec{r}_1 d\vec{\rho}_1) \ \ldots \int_0^t (d\tau_n d\vec{r}_n d\vec{\rho}_n) \ * \tag{B-4}$$

$$T\left[\gamma(\vec{r}_1 - \vec{\rho}_1)\delta\left(\vec{r}_T(\omega,\tau_1) - \vec{r}_1\right) \ \delta\left(\vec{r}_S(\sigma,\tau_1) - \vec{\rho}_i\right) \ldots\right.$$

$$\left.\ldots \ \gamma(\vec{r}_n - \vec{\rho}_n) \ \delta\left(\vec{r}_T(\omega,\tau_n) - \vec{r}_n\right) \ \delta\left(\vec{r}_S(\sigma,\tau_n) - \vec{\rho}_n\right)\right] \ .$$

Hence,

$$\bar{p}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^t (d\tau_1 d\vec{r}_1 d\vec{\rho}_1) \ \ldots \ (d\tau_n d\vec{r}_n d\vec{\rho}_n) \ *$$

$$T\Big[\gamma(\vec{r}_1 - \vec{\rho}_1) \ \ldots \ \gamma(\vec{r}_n - \vec{\rho}_n) \int d\omega d\sigma \ A(\sigma,\omega) \ * \tag{B-5}$$

$$\delta\Big(\vec{r}_T(\omega,\tau_1) - \vec{r}_1\Big)\delta\Big(\vec{r}_S(\sigma,\tau_1) - \vec{\rho}_1\Big) \ \ldots \ \delta\Big(\vec{r}_T(\omega,\tau_n) - \vec{r}_n\Big)\delta\Big(\vec{r}_S(\sigma,\tau_n) - \vec{\rho}_n\Big)\Big],$$

where $A(\sigma,\omega)$ stands for the joint probability that the searcher follow path σ and the target path ω.

We shall assume for simplicity that:

$$A(\sigma,\omega) = A_1(\sigma) \ A_2(\omega) \tag{B-6}$$

which is tantamount to ignoring the dynamics of the search and detection game. Correspondingly,

$$\bar{p}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^t (d\tau_1 d\vec{r}_1 d\vec{\rho}_1) \ \ldots \int_0^t (d\tau_n d\vec{r}_n d\vec{\rho}_n) \ * \tag{B-7}$$

$$T\left[\prod_{i=n}^{n} \gamma(\vec{r}_i - \vec{\rho}_i) \ G_T\Big(\vec{r}_i,\tau_i \big| \vec{r}_{i-1},\tau_{i-1}\Big) \ G_S\Big(\vec{\rho}_i,\tau_i \big| \vec{\rho}_{i-1},\tau_{i-1}\Big)\right] .$$

Again, we i itroduce:

$$\psi_n\left(\vec{r}_{n+1},\vec{\rho}_{n+1},\tau_{n+1} \mid \vec{r}_0,\vec{\rho}_0,\tau_0\right) \equiv$$

$$\int_0^t (d\tau_1 d\vec{r}_1 d\vec{\rho}_1) \ldots (d\tau_n d\vec{r}_n d\vec{\rho}_n)\, G_T\left(\vec{r}_{n+1},\tau_{n+1} \mid \vec{r}_n,\tau_n\right) G_s\left(\vec{\rho}_{n+1},\tau_{n+1} \mid \vec{\rho}_n,\tau_n\right) * \tag{B-8}$$

$$\prod_{i=1}^{n} \gamma(\vec{r}_i-\vec{\rho}_i)\; G_T\left(\vec{r}_i,\tau_i \mid \vec{r}_{i-1},\tau_{i-1}\right) G_s\left(\vec{\rho}_i,\ {}_i \mid \vec{\rho}_{i-1},\tau_{i-1}\right) ,$$

to write,

$$\bar{p}\ (t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^t (d\tau_n d\vec{r}_n d\vec{\rho}_n)\ \gamma(\vec{r}_n-\vec{\rho}_n)\ \psi_{n-1}(\vec{r}_n,\vec{\rho}_n,\tau_n \mid \vec{r}_0,\vec{\rho}_0,\tau_0) \tag{B-9}$$

and notice that ψ_n satisfy the iterative equation, (see figure 6).

$$\psi_n\left(\vec{r}_{n+1},\vec{\rho}_{n+1},\tau_{n+1} \mid \vec{r}_0,\vec{\rho}_0,\tau_0\right) =$$

$$\int_0^{\tau_{n+1}} (d\tau_n d\vec{r}_n d\vec{\rho}_n)\ G_s\left(\vec{\rho}_{n+1},\tau_{n+1} \mid \vec{\rho}_n,\tau_n\right) G_T\left(\vec{r}_{n+1},\tau_{n+1} \mid \vec{r}_n,\tau_n\right) * \tag{B-10}$$

$$\gamma\left(\vec{r}_n-\vec{\rho}_n\right) \psi_{n-1}\left(\vec{r}_n,\vec{\rho}_n,\tau_n \mid \vec{r}_0,\vec{\rho}_0,\tau_0\right) .$$

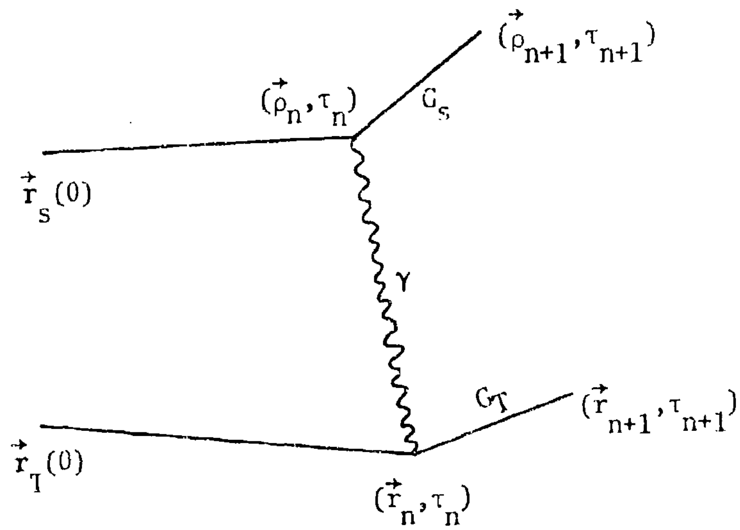

FIG. 6: GEOMETRY OF THE ITERATIVE SOLUTION FOR GLOBAL DETECTION PROBABILITIES

Naturally, equation (B-10) contains physical information similar to that identified in equation (A-10) and therefore most of the discussion following equation (A-10) should apply here too. Also, by the nature of (B-10), some generalization of the Monte Carlo methods suggested there for the solution to the specified search pattern problem ought to be useful in finding global probabilities of detection.

DISCUSSION

Motivated by the need for developing a framework of larger breadth and flexibility than the Koopman one, we have succeeded in delineating a train of thought that holds the promise of successfully accounting for the modern sophistication of the search and detection game.

Thus, a principle of biasing has been employed, whereby given probabilities are assigned to all possible paths of a military unit, and the probability of arrival of that unit at any point in the ocean expressed in terms of them. It then becomes possible, as shown in Section C, to formulate the result in terms of iterative solutions whose physical content is interestingly simple and whose calculational advantages are many fold. In fact, a full solution can be obtained numerically, if the input probabilities per path are given. Whether they reflect our total ignorance concerning the unit's motion or the highly correlated nature of the path, a choice of such inputs is always possible, in that the inputs merely represent our actual data base and not some restrictive model assumption that might not correspond to the reality we want to describe. The flexibility that this imparts to the formalism is such that one might approach within the same framework problems that have varying amounts of input information.

Let us also note, that by the very nature of the iterative solutions we propose, a connection can be established between the search and detection problem on one hand and the well studied mathematical theory of Green's function on the other, thus opening a door for the influx of a whole family of numerical methodologies that have proven useful elsewhere. Although much might still be said about the hidden or obvious dimensions of this framework, we stop our discussion here for we wish merely to enlighten our reader's judgement, not to influence it.